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Electron Spin Transition Solution Applicable to an Ensemble of Isolated Electrons[1]

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An electron spin aligned with a static magnetic field changes its orientation when subjected to a time-varying magnetic field which is directed perpendicular to the static magnetic field. This well-known phenomenon is readily calculated when the time-varying magnetic field is circularly polarized; however, the evolution of the spin-state wavefunctions becomes much more difficult to calculate when the time-varying magnetic field is linearly polarized. For linear polarization and isolated spins, an analytic solution has been derived for the dynamical spin-state wavefunctions. Part of the solution procedure relies on an expansion using a small parameter, which is the ratio of the amplitude of the time-varying magnetic field to the static magnetic field. To verify the validity of the expansion technique, a numerical solution of the basic equations is compared to the analytic solution. Results are found to agree to better than 10% for exact resonance and better than 5% in general. © 1999 Academic Press

Key Words: **electron spin resonance; ESR; electron spin transition; analytic electron spin solution; fermion spin transition; rotating wave approximation.**

1. INTRODUCTION

Properties of matter have been studied for decades using electromagnetic resonances. One of the typical examples of this endeavor is the study of properties of matter utilizing the transition between two energy levels. Early work in this area was done by Feynman (*1*) in which it was shown that a geometrical representation can be ascribed to the two-level Schrödinger equation. For magnetic interaction with a spin-$\frac{1}{2}$ system, he demonstrated that the three-dimensional geometric representation reduces to physical space. This means techniques developed for the spin-$\frac{1}{2}$ system can be adapted to other two-level problems, where the three dimensions are no longer in general physical space. In other words, the classical vector model for spin precession applies to any two-level transition, subject only to several restrictions on the interaction matrix elements. Consistent with this viewpoint, Abragam (*2*) describes the general problem of two levels coupled by an RF field, where he introduces by analogy the notion of fictitious spin and fictitious magnetic field. Several of his examples are transitions between two spin levels influenced by a small quadrupole interaction, and the problem of the populations of two states that include electronic and nuclear spin. A two-level atom is considered by Levenson (*3*) for the application of laser spectroscopy. He derives a master equation for the transition between levels in the density matrix formulation which reduces (upon setting $\rho_{11} = \rho_{22} = 0$) to the mathematically equivalent set of equations that will be applied later to electron spin. Letokhov and Chebotayev (*4*) consider an isolated two-level system ignoring relaxation and damping and again derives the mathematically equivalent set of equations later applied to electron spin. Any system that has the mathematically equivalent set of equations as those for electron spin will also have the same solution. It is only necessary to interpret the electron spin solution in terms of nuclear spin, a sum of electron and nuclear spins, or elements of a density matrix.

The general physical model under consideration in this work is a two-level system immersed in a static magnetic field and also subject to a perpendicular time-varying field. The perpendicular time-varying magnetic field causes transitions between the two energy levels. In particular, for particles with spin $\frac{1}{2}$ such as an electron, there can only be two possible spin states with respect to a direction in space defined by a static magnetic field. In either of these states, when subject to a time-varying perpendicular magnetic field, the electron spin can change its orientation. The time dependence of this behavior is readily calculated for a time-varying, circularly polarized perpendicular field as shown by Rabi (*5*). When the perpendicular field is linearly polarized, the evolution of the state occupations becomes much more difficult to calculate. Due to the relative ease of solving the circularly polarized case (*6*), many researchers (*7–17*) use the rotating-wave approximation (RWA). The RWA is not invoked when a system is modeled with a driver that is truly circularly polarized (*5*, *6*). The RWA is used when the time-varying perpendicular field is truly linearly polarized and it is represented as being circularly polarized.

A linearly polarized field can be decomposed into two circularly polarized fields, rotating in opposite directions. Because the response to the two circularly polarized fields can be considered separately, there are two common justifications for the RWA. The first justification is that only one of the two circularly polarized fields is strongly interacting near resonance, and the other circularly polarized field can be neglected (*1*). This is reasonable

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because the spin precesses around the direction established by the static magnetic field. Only one of the circularly polarized fields rotates in the precession direction. The second justification relies on time scales. As explained by Gauthey *et al.* (*18*), the RWA is valid if the interaction term is averaged over a time much longer than the period of the time-varying field, but much shorter than the state evolution time.

The RWA is useful; however, it is not universally applicable. The experiments of Dodd *et al.* (*19*) are an extreme situation where the ratio of perpendicular time-varying magnetic field to the static field is varied from a small value all the way up to 4. For small values of the ratio, Dodd found that the RWA was satisfactory, but at large values he observed that the spectrum changed dramatically. In early work, Bloch and Siegert (*20*) showed the resonant frequency shifts when the full field is considered, rather than just the RWA. While investigating the Bloch–Siegert shift, Wei *et al.* (*21*) demonstrated that the Autler–Townes energy level splitting (*22*) is asymmetric and that the asymmetry can be attributed to the inclusion of the full driver field, rather than just using the RWA. The RWA in this case incorrectly predicts that the energy level splitting is perfectly symmetric. In a study of a single mode laser, Vyas and Singh (*23*) found both transient and secular effects when their analysis included the full linear polarization, rather than the RWA. In a study of a related set of three coupled equations, Milonni *et al.* (*24*) showed that a two-level atom interacting with an electric field can exhibit chaotic behavior. He further determined that the RWA of the same set of equations does not manifest chaos. Drummond (*25*) found he could not use the RWA since he was concerned with detector response for ultrafast lasers. He concluded that, on a time scale of a few cycles, the RWA becomes completely invalid. Fang and Zhou (*26*) discovered that the combination of both circularly polarized fields played an important role in the solution for quantum entropy evolution. The RWA left out high-frequency amplitude modulation of the quantum entropy evolution. Clearly, all of these examples demonstrate that the RWA may inadvertently conceal physics which is revealed when the RWA is not used.

In the derivation of spin-state dynamics that follows, the time-varying perpendicular field is truly linearly polarized and no RWA is made. This derivation is different than previous derivations in several respects. A Hamiltonian that is periodic in time was investigated by Shirley (*27*), which yielded a set of equations identical in form to those applicable to the electron spin problem. However, in his work, Floquet's theorem was invoked to express the solution of Schrödinger's equation with a Fourier series expansion leading to an infinite matrix. He approximated the infinite matrix as a finite dimension matrix and from this form he derived eigenvalues and eigenfunctions. In this work, Floquet's theorem is not invoked, and no Fourier series decomposition is made.

Using another approximation, the equations addressed in this work were solved in a general model employed by Bloch and Siegert (*20*) in which the solution for probability amplitudes is written for arbitrary elliptical polarization. This includes linear polarization of the time-varying magnetic field as a limiting case. Their solution is written as an expansion in a parameter that depends on the magnitude of the deviation of the driver frequency from resonance and also depends on the magnitude of the elliptically polarized field. They also solve for a function that determines the ratio of the probability amplitudes, rather than the individual probability amplitudes. In this work, a solution is derived for the individual probability amplitudes separately. This procedure avoids difficulties related to the fact that the ratio of probability amplitudes is singular when every spin is definitely in one state. The solution derived in this work relies on an expansion parameter that only depends on the ratio of the magnitude of the time-varying magnetic field to the magnitude of the static magnetic field. There is no linkage of the frequency of the time-varying field to the expansion parameter.

In many circumstances, electron beams are transported in a magnetic field produced by solenoids. The primary direction of the magnetic field produced by a solenoid defines the z axis. In general then, a transported electron will have a spin that is in one of two possible states associated with the z direction defined by the magnetic field produced by a solenoid. If, in addition, there is a time-varying magnetic field perpendicular to the z direction, an electron undergoes transitions between these two states. Possible experimental arrangements consist of applying the perpendicular magnetic field with a linearly polarized RF field, or an oscillating dipole magnet, which also corresponds to the case of linear polarization. It is desirable to diagnose electron particle beams which are immersed in a magnetic field without disturbing the transport of the beam. One of the attributes of the beam that can be examined with minimal impact on the transport is the electron spin. The spin-state wavefunction response to the perpendicular diagnostic magnetic field can be related to beam properties such as density and beam size. In this work, a time-dependent solution has been derived for the spin-state probability amplitudes as a first necessary step toward understanding how the electron spin varies when subjected to a perpendicular time-varying magnetic field.

In Section 2, the basic spin-state equations are derived as a set of two coupled first-order equations. The derivative in each of the coupled equations is proportional to its own state wavefunction and a linear coupling term proportional to the other state with a multiplier which scales like the ratio of the amplitude of the perpendicular magnetic field to the z-directed magnetic field. Because the perpendicular time-varying magnetic field is assumed to be much smaller than the steady z-directed magnetic field, the coupling effect can be viewed as a perturbation. The linearly polarized perpendicular magnetic field can be considered to be composed of two parts. The first part has positive phase advance as time increases, and the second part has negative phase advance as time increases. The source of the difficulty in obtaining the time dependence of the two spin states is illustrated in Section 3, where an analytic solution is derived for the positive phase advance part of the linearly polarized field. Even though the positive phase advance field solution can also be derived for the negative phase advance field, due to the coupling, it is not possible to obtain the total perpendicular field solution from any simple combination of these two solutions. To derive the desired solution with a linearly applied perpendicular magnetic field, it is first necessary to formulate the spin equation in new variables, as explained in Section 4. The actual perturbation solution for each state along with the resonance solution is presented in Sections 5, 6, and 7. The analytic perturbation solutions are compared to the numerical solution of the original equation in Section 8. Good agreement between the numerical and analytic solutions is obtained, which demonstrates the validity of the assumptions used to derive the analytic perturbation solution. In Section 9, the derived analytic equations are compared to solutions from previous research.

2. FORMULATION OF THE SPIN EQUATION

An isolated spin system characteristic of a multiampere electron beam interacting with a magnetic field is considered. An estimate of the corresponding number of electrons in this model can be made assuming a constant current density. For a beam radius a, velocity $\hat{v} = v/c$, and current I the relation between current density and charge density ρ is $J = I/(\pi a^2) = \rho \hat{v} c$, where c is the speed of light in vacuum. Rearranging this expression, the electron number per volume is then $\rho/e = 6.63 \times 10^{13}(I/\hat{v})(0.01/a)^2$. For a radius of 0.01 m and a current of 1 amp, the number per volume is $6.63 \times 10^{13}/\hat{v}$ m^{-3}. Each spin interacts with the external applied magnetic field; however, there is a question of whether or not there is a spin–spin interaction. This question is answered by comparing the size of the external magnetic field to the magnetic field produced by the magnetic moment of an electron. The magnetic field of the electron can be estimated by the ratio of the magnetic moment of the electron to the cube of a separation distance between electrons. The inverse cube root of the number per volume gives an approximation of the separation between electrons, $L = 2.47 \times 10^{-5}(\hat{v}/I)^{1/3}(a/0.01)^{2/3}$. Since $\hat{v}$ is always less than 1, a reasonable approximation is $L = 2.47 \times 10^{-5}$. The electron magnetic field estimate is then $B^{\text{electron}} = (he/4\pi m)/L^3 = 6.16 \times 10^{-10}$. A typical external magnetic field value would range from 0.01 to 0.30 T, and therefore it would be estimated to be about 10^9 larger than the electron magnetic field. Thus the governing equation does not include spin–spin interaction terms.

The configuration under study is one of a steady-state magnetic field in the z direction, in combination with a time-varying driver field in the x direction. Using spin matrices, the time-dependent Schrödinger equation (*28*) becomes

$$i \frac{\partial}{\partial t}\begin{bmatrix}\Psi_1\\ \Psi_2\end{bmatrix} = \frac{e}{2m}\left(B_0\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix} + B_1 \cos \omega t \begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}\right)\begin{bmatrix}\Psi_1\\ \Psi_2\end{bmatrix}, \quad [1]$$

where B_0 is the magnitude of the steady-state magnetic field in the z direction, B_1 is the magnitude of the perpendicular time-varying magnetic field with frequency ω, e is the magnitude of the electron charge, m is the electron mass, and $i^2 = -1$. The functions Ψ_1 and Ψ_2 are the probability amplitudes of the two states. Under the condition that the spins are in state 1 at a particular time, then $|\Psi_2|^2$ would be the transition probability to state 2. The matrix equation in Eq. [1] can be written as a system of two simultaneous equations,

$$\frac{d\Psi_1}{d\tau} = -i\Psi_1 - i\left(\frac{\beta}{2}\right)\Psi_2 \cos \alpha\tau$$

$$\frac{d\Psi_2}{d\tau} = i\Psi_2 - i\left(\frac{\beta}{2}\right)\Psi_1 \cos \alpha\tau, \quad [2]$$

where $\omega_0 = eB_0/(2m)$, $\omega_1 = eB_1/m$, $\alpha = \omega/\omega_0$, $\beta = \omega_1/\omega_0$, and $\tau = \omega_0 t$. Note that because $B_1 \ll B_0$ the $\cos \alpha\tau$ term in Eq. [2], which is order unity, is multiplied by a parameter β which is small compared to the order unity first term on the right. To maintain consistency with the exclusion of spin–spin interaction in Eq. [1], $\beta < 1$; however, $\beta \geq 10^{-6}$, so it dominates the size of the spin–spin interaction. The complexity involved in the solution of Eq. [2] is due to the fact that the cos $\alpha\tau$ term can be considered to be a sum of two time-varying drives which have positive and negative phase advance. Thus a transform into either the negative phase or the positive phase advance frame can eliminate only one of the two. Having both phases present, which is the situation for linear polarization, results in one direction coinciding with the precession direction of the electron spin and thus interacting strongly. The opposite phase advance component only interacts weakly.

3. SOLUTION FOR ONLY ONE DIRECTION OF PHASE ADVANCE IN THE DRIVER

By definition, $\cos \alpha\tau \equiv (e^{i\alpha\tau} + e^{-i\alpha\tau})/2$ and thus as discussed, Eq. [2] includes both directions of phase advance. To determine the solution resulting from the effect of just the positive phase advance part, $e^{i\alpha\tau}$, of the complete perturbation term, for the moment consider a revised set of equations, with only one part of the cosine included:

$$\frac{d\Psi_1}{d\tau} = -i\Psi_1 - i\left(\frac{\beta}{2}\right)\Psi_2 e^{i\alpha\tau}$$

$$\frac{d\Psi_2}{d\tau} = i\Psi_2 - i\left(\frac{\beta}{2}\right)\Psi_1 e^{i\alpha\tau}. \quad [3]$$

The form of Eq. [3] is not the RWA because a y direction field multiplying the y spin matrix was excluded from Eq. [1]. Only the x component of the perpendicular field has been included and thus the decomposition that is in view in Eq. [3] is solely of the $\cos \alpha\tau$ function. After substituting new variables, $\varphi_1 \equiv \Psi_1 e^{i\tau}$ and $\varphi_2 \equiv \Psi_2 e^{-i\tau}$, Eq. [3] becomes

$$\frac{d\varphi_1}{d\tau} = -i\left(\frac{\beta}{2}\right)\varphi_2 e^{i(\alpha+2)\tau}$$

$$\frac{d\varphi_2}{d\tau} = -i\left(\frac{\beta}{2}\right)\varphi_1 e^{i(\alpha-2)\tau}. \quad [4]$$

An equation for φ_1 is obtained by taking the derivative of the first equation in Eq. [4] and changing the dependent variable to $\mu \equiv e^{i\alpha\tau}$,

$$\frac{d^2\varphi_1}{d\mu^2} - \left(\frac{2}{\alpha\mu}\right)\frac{d\varphi_1}{d\mu} - \left(\frac{\beta}{2\alpha}\right)^2 \varphi_1 = 0. \qquad [5]$$

Equation [5] has the form of the differential equation for Bessel functions and has as its solution the product of a power of μ with a Bessel function $J_\kappa((i\beta/2\alpha)\ \mu)$. Thus,

$$\varphi_1 = \mu^\kappa J_\kappa, \qquad [6]$$

where $\kappa \equiv (\alpha + 2)/(2\alpha)$. The order of the Bessel function and power of μ are determined by the selection of frequency ratio α. In like manner, a solution similar to Eq. [6] can be obtained by changing $e^{i\alpha\tau}$ to $e^{-i\alpha\tau}$ in Eq. [3] and again deriving an equation like Eq. [5]. Unfortunately, no combination of functions having the form of such solutions can be made into a solution of Eq. [2], because $\cos\alpha\tau$ can be written as a sum of two exponentials $(e^{i\alpha\tau} + e^{-i\alpha\tau})/2$, and both exponentials appear simultaneously in Eq. [2]. However, the solvable form of Eq. [4] gives some insight into a useful variable substitution, and the appropriate variable substitution leads ultimately to a tractable solution.

4. FORMULATION OF THE SPIN EQUATION IN NEW VARIABLES

Insight into the form of the desired solution is provided by the integral representation of the Bessel function appearing in Eq. [6]. For example, if $\alpha = 2$ then $\kappa = 1$ and $J_1(p) = (1/\pi)\int_0^\pi \cos(p\sin\theta - \theta)d\theta$. Since argument p depends on μ and cosine can be written as the sum of two exponentials, it can be seen from Eq. [6] that the total solution with both directions of phase advance is likely to consist of a form having an exponential of a function of μ. Furthermore, the sum and difference of the original spin states provides a view into the two phase directions of the decomposed perpendicular magnetic field. Thus after attempting many possibilities based on these ideas, new variables y_1 and y_2 are introduced, with definitions consisting of a combination of the original variables.

$$y_1 = (\Psi_1 + \Psi_2)e^{i(\beta/2\alpha)\sin\alpha\tau}$$
$$y_2 = (\Psi_1 - \Psi_2)e^{-i(\beta/2\alpha)\sin\alpha\tau} \qquad [7]$$

The differential equations satisfied by y_1 and y_2 are

$$\frac{dy_1}{d\tau} = -iy_2 e^{i(\beta/\alpha)\sin\alpha\tau}$$
$$\frac{dy_2}{d\tau} = -iy_1 e^{-i(\beta/\alpha)\sin\alpha\tau}. \qquad [8]$$

For small τ, the form of Eq. [8] is reminiscent of Eq. [4] since $(\sin\alpha\tau)/\alpha \approx \tau$; however, the sign in the exponentials is plus and minus and consequently the early time solution is a sum of sinusoids with frequencies $[\beta + \sqrt{\beta^2+4}]/2$ and $[\sqrt{\beta^2+4} - \beta]/2$.

For general values of the time, a derivative of the first equation in Eq. [8] is taken to form an equation only involving y_1. The basic equation to be solved is then

$$\frac{d^2y_1}{d\tau^2} - (i\beta\cos\alpha\tau)\frac{dy_1}{d\tau} + y_1 = 0. \qquad [9]$$

An interesting insight into the solution of Eq. [9] can be obtained by making a substitution of $y_1 = q(\tau)e^{i(\beta/2\alpha)\sin\alpha\tau}$, which transforms Eq. [9] into an equation for the variable $q(\tau)$, resulting in a form very similar to Hill's equation (*29*) in his Section 18.22, Eq. [1] and Eq. [2],

$$\frac{d^2q}{d\tau^2} + (\tfrac{1}{8})(8 + \beta^2 + \beta^2\cos(2\alpha\tau) - 4i\beta\alpha\sin\alpha\tau)q = 0. \qquad [10]$$

Hill's equation is a Mathieu-like equation which arises in the study of the lunar perigee problem. It can be seen from Eq. [10] that the behavior is like an oscillator with a frequency that has a small dependence on τ. This leads to the conclusion that ultimately the solution must involve all harmonics of $e^{i\alpha\tau}$. Furthermore, it conveys the expectation that the solution y_1 consists of a nearly harmonic oscillator solution modulated by $e^{i(\beta/2\alpha)\sin\alpha\tau}$.

5. PERTURBATION SOLUTION FOR y_1

The oscillator form of Eq. [10] with $\beta \ll 8$ has a small time variation of the frequency, which suggests that the solution of Eq. [9] can be obtained using perturbation methods with a smallness parameter of β. Because the perpendicular magnetic field is always less than the static field, it is always true that $\beta < 1$. To more succinctly represent the perturbation solution and also avoid secularities, Eq. [9] is converted to a driven Riccati equation (*30*), Eq. [1.1], by making the substitution, $\log(y_1) = \log(\bar{C}_{11}) - i\int u_1 d\tau$, where $\bar{C}_{11}$ is a constant. This relation produces a first-order nonlinear equation,

$$\frac{du_1}{d\tau} - (i\beta\cos\alpha\tau)u_1 - iu_1^2 = -i. \qquad [11]$$

The spin solution y_1 is obtained by solving Eq. [11] for u_1 first, then integrating with respect to τ, and finally exponentiating. Assuming that a particular solution can be found, the solution of Eq. [11] can be written as a sum of two functions $u_1 = u_{p1} + 1/v_1$, where the first function, u_{p1}, is a particular solution of Eq. [11]. Substituting into Eq. [11] gives the equation

$$\frac{dv_1}{d\tau} + (i2u_{p1} + i\beta\cos\alpha\tau)v_1 = -i, \qquad [12]$$

which must be solved by v_1. This is just a first-order differential equation with solution

$$v_1 = \bar{C}_{12}e^{-P_1} - ie^{-P_1}\int e^{P_1}d\tau, \qquad [13]$$

where $\bar{C}_{12}$ is a constant. For convenience, define $Q_1(\tau) = -i \int u_{p1} d\tau$ and then

$$P_1(\tau) = i(\beta/\alpha)\sin(\alpha\tau) - 2Q_1(\tau). \quad [14]$$

The arbitrary constants $\bar{C}_{11}$ and $\bar{C}_{12}$ provide the flexibility needed to specify the initial value and derivative of y_1. Using Eq. [13], the reciprocal function can be expressed as

$$\frac{1}{v_1} = i\frac{d}{d\tau}\left(\log\left[\bar{C}_{12} - i\int e^{P_1} d\tau\right]\right), \quad [15]$$

which, when substituted into the definition of y_1, leads to a modified expression for the solution

$$y_1 = e^{Q_1}\left[C_{11} - iC_{12}\int e^{P_1} d\tau\right], \quad [16]$$

with new coefficients defined in terms of the original arbitrary coefficients, $C_{11} = \bar{C}_{11}\bar{C}_{12}$ and $C_{12} = \bar{C}_{11}$. The complete solution of Eq. [11] is determined if a particular solution, u_{p1}, can be found, since this is the function that is fundamental to the determination of $P_1(\tau)$ and $Q_1(\tau)$. A particular perturbation solution of Eq. [11] can be determined at each order in β, by assuming

$$u_{p1} = 1 + \beta f_1 + \beta^2 f_2 + \beta^3 f_3 + \cdots, \quad [17]$$

where only terms up to third order in β are kept. It is clear that the series could continue out to any desired order with a consequent increase in complexity. It is found at order β, the equation that must be solved is

$$\frac{df_1}{d\tau} - i\cos\alpha\tau - 2if_1 = 0. \quad [18]$$

Equation [18] is a first-order equation that can be readily solved yielding

$$f_1 = D_1\cos\alpha\tau + iD_2\sin\alpha\tau, \quad [19]$$

where

$$D_1 = \frac{2}{\alpha^2 - 4}$$

$$D_2 = \frac{\alpha}{\alpha^2 - 4}. \quad [20]$$

It is required that $\alpha \neq 2$ to prevent a singular solution; the situation of $\alpha = 2$ which is the resonance condition $\omega = 2\omega_0$ will be discussed later in Section 7. The equation that must be solved at order β^2 is

$$\frac{df_2}{d\tau} - i(\cos\alpha\tau) f_1 - 2if_2 - if_1^2 = 0, \quad [21]$$

which is an analytically solvable first-order equation having solution

$$f_2 = D_3 + D_4\cos 2\alpha\tau + iD_5\sin 2\alpha\tau, \quad [22]$$

where

$$D_3 = -\frac{1}{4(\alpha^2 - 4)}$$

$$D_4 = \frac{4 + \alpha^2}{4(\alpha^2 - 4)^2}$$

$$D_5 = \frac{\alpha}{(\alpha^2 - 4)^2}. \quad [23]$$

The equation that must be solved at order β^3 is

$$\frac{df_3}{d\tau} - i(\cos\alpha\tau) f_2 - 2if_3 - 2if_1 f_2 = 0, \quad [24]$$

which is again an analytically solvable first-order equation having solution

$$f_3 = D_6\cos\alpha\tau + D_7\cos 3\alpha\tau + iD_8\sin\alpha\tau + iD_9\sin 3\alpha\tau, \quad [25]$$

where

$$D_6 = \frac{9\alpha^4 - 4\alpha^2}{(\alpha^2 - 4)^2(80\alpha^2 - 32 - 18\alpha^4)}$$

$$D_7 = \frac{-12\alpha^2 - 5\alpha^4}{(\alpha^2 - 4)^2(80\alpha^2 - 32 - 18\alpha^4)}$$

$$D_8 = \frac{32\alpha^3 - 16\alpha + 9\alpha^5}{(\alpha^2 - 4)^2(320\alpha^2 - 128 - 72\alpha^4)}$$

$$D_9 = \frac{-16\alpha - 48\alpha^3 - 3\alpha^5}{(\alpha^2 - 4)^2(320\alpha^2 - 128 - 72\alpha^4)}. \quad [26]$$

In order that coefficients D_7, D_8, and D_9 remain finite, $\alpha \neq \frac{2}{3}$. Now knowing the particular solution from the definition of Q_1, it is found that

$$\begin{aligned} Q_1(\tau) &= -i\int (1 + \beta f_1 + \beta^2 f_2 + \beta^3 f_3) d\tau \\ &= -iD_{00}\tau - \left(\frac{\beta}{\alpha}\right)\Big[iD_1\sin\alpha\tau + D_2\cos\alpha\tau \\ &\quad + i\beta\frac{D_4}{2}\sin 2\alpha\tau + \beta\frac{D_5}{2}\cos 2\alpha\tau \\ &\quad + i\beta^2 D_6\sin\alpha\tau + \tfrac{1}{3}i\beta^2 D_7\sin 3\alpha\tau \\ &\quad + \beta^2 D_8\cos\alpha\tau + \tfrac{1}{3}\beta^2 D_9\cos 3\alpha\tau\Big], \end{aligned} \quad [27]$$

where $D_{00} = 1 + \beta^2 D_3$. The Eq. [27] result can be inserted into Eq. [16], which leaves only one integral to be determined to obtain the entire solution. The integration is performed by breaking $P_1(\tau)$ into two integrand parts as

$$-i\int e^{P_1}d\tau = -i\int e^{2iD_{00}\tau}[e^{i(\beta/\alpha)\sin\alpha\tau - 2Q_1 - 2iD_{00}\tau}]d\tau. \qquad [28]$$

The function in the square bracket in Eq. [28] is expanded to third order in β, and the integration is performed. This procedure is the key to avoiding an expansion in terms of Bessel functions leading to secularities and intractable integrals. The solution is obtained upon integrating Eq. [28] and substituting into Eq. [16]

$$\begin{aligned} y_1 = C_{11}e^{Q_1} + C_{12}e^{Q_1+2iD_{00}\tau}[&-\tfrac{1}{2} + \beta A_1\cos\alpha\tau \\ &+ i\beta A_2\sin\alpha\tau + \beta^2 A_3 + \beta^2 A_4\cos 2\alpha\tau \\ &+ \beta^2 A_5\sin 2\alpha\tau + \beta^3 A_6\cos\alpha\tau + \beta^3 A_7\cos 3\alpha\tau \\ &+ \beta^3 A_8\sin\alpha\tau + \beta^3 A_9\sin 3\alpha\tau]. \end{aligned} \qquad [29]$$

In Eq. [29], every term in the square bracket is bounded, and the real part of $Q_1(\tau)$ consists solely of trigonometric functions, which means as $\tau \to \infty$ the magnitude of y_1 is bounded. Consequently, y_1 is normalizable. Also as a check, in the limit as $\beta \to 0$, in Eq. [29] the proper form of the solution is recovered compared to the solution of Eq. [9] when β is set to zero, $y_1 = C_{11}e^{-i\tau} - (\frac{1}{2})C_{12}e^{i\tau}$. The first-order coefficients in Eq. [29] are defined as

$$\begin{aligned} A_1 &= \frac{D_0}{4-\alpha^2} - \frac{4D_2}{\alpha(4-\alpha^2)} \\ A_2 &= \frac{2D_2}{4-\alpha^2} - \frac{2D_0}{\alpha(4-\alpha^2)}, \end{aligned} \qquad [30]$$

where $D_0 = 1 + 2D_1$. The coefficients in Eq. [29] that are second order in β are

$$\begin{aligned} A_3 &= \frac{D_0^2}{8\alpha^2} - \frac{D_2}{2\alpha^2} + \frac{D_3}{2} \\ A_4 &= \frac{1}{8-8\alpha^2}\left[\frac{4D_0D_2}{\alpha} - \frac{D_0^2}{\alpha^2} - \frac{4D_2^2}{\alpha^2} + 4D_4 - \frac{4D_5}{\alpha}\right] \\ A_5 &= \frac{1}{8-8\alpha^2}\left[\frac{D_0^2}{\alpha} - \frac{4D_0D_2}{\alpha^2} + \frac{4D_2^2}{\alpha} - \frac{4D_4}{\alpha} + 4D_5\right]. \end{aligned} \qquad [31]$$

In order for the A_4 and A_5 coefficients to remain finite $\alpha \neq 1$. The coefficients in Eq. [29] that are third order in β are

$$\begin{aligned} A_6 = \frac{1}{4-\alpha^2}\Bigg[&\frac{D_0^2D_2}{2\alpha^3} - \frac{D_0^3}{8\alpha^2} + \frac{D_0D_2^2}{2\alpha^2} - \frac{8D_0D_3}{4-\alpha^2} \\ &+ \frac{32D_2D_3}{\alpha(4-\alpha^2)} - \frac{4D_2D_3}{\alpha} - \frac{2D_2^3}{\alpha^3} + \frac{D_0D_4}{\alpha^2} \\ &+ \frac{D_2D_4}{\alpha} - \frac{D_0D_5}{2\alpha} - \frac{2D_2D_5}{\alpha^2} + 2D_6 - \frac{4D_8}{\alpha}\Bigg] \\ A_7 = \frac{1}{96-216\alpha^2}\Bigg[&\frac{3D_0^3}{\alpha^2} - \frac{12D_0^2D_2}{\alpha^3} + \frac{36D_0D_2^2}{\alpha^2} - \frac{16D_2^3}{\alpha^3} \\ &- \frac{24D_0D_4}{\alpha^2} + \frac{72D_2D_4}{\alpha} + \frac{36D_0D_5}{\alpha} \\ &- \frac{48D_2D_5}{\alpha^2} + 48D_7 - \frac{32D_9}{\alpha}\Bigg] \\ A_8 = \frac{1}{4-\alpha^2}\Bigg[&\frac{D_0^3}{4\alpha^3} - \frac{D_0^2D_2}{4\alpha^2} - \frac{D_0D_2^2}{\alpha^3} + \frac{D_2^3}{\alpha^2} - \frac{16D_2D_3}{4-\alpha^2} \\ &+ \frac{16D_0D_3}{\alpha} - \frac{2D_0D_3}{\alpha} - \frac{D_0D_4}{2\alpha} - \frac{2D_2D_4}{\alpha^2} \\ &+ 2D_8 + \frac{D_0D_5}{\alpha^2} + \frac{D_2D_5}{\alpha} - \frac{4D_6}{\alpha}\Bigg] \\ A_9 = \frac{1}{48-108\alpha^2}\Bigg[&\frac{9D_0^2D_2}{\alpha^2} - \frac{D_0^3}{\alpha^3} - \frac{12D_0D_2^2}{\alpha^3} + \frac{12D_2^3}{\alpha^2} \\ &+ \frac{18D_0D_4}{\alpha} - \frac{24D_2D_4}{\alpha^2} + 24D_9 - \frac{12D_0D_5}{\alpha^2} \\ &+ \frac{36D_2D_5}{\alpha} - \frac{16D_7}{\alpha}\Bigg]. \end{aligned} \qquad [32]$$

6. PERTURBATION SOLUTION FOR y_2

In order to obtain y_2, a second-order equation similar to Eq. [9] must be solved.

$$\frac{d^2y_2}{d\tau^2} + (i\beta\cos\alpha\tau)\frac{dy_2}{d\tau} + y_2 = 0. \qquad [33]$$

It is possible to proceed as described previously in Eqs. [11]–[32] to obtain y_2. By making the substitution, $\log(y_2) = \log(\bar{C}_{21}) - i\int u_2 d\tau$, Eq. [33] is converted to a driven Riccati equation, where $\bar{C}_{21}$ is a constant and this relation then gives

$$\frac{du_2}{d\tau} + (i\beta\cos\alpha\tau)u_2 - iu_2^2 = -i. \qquad [34]$$

The solution for y_2 is obtained by solving Eq. [34] for u_2 first, then integrating with respect to τ, and finally exponentiating. The solution of Eq. [34] can be written as a sum of two functions $u_2 = u_{p2} + 1/v_2$, where the first function, u_{p2}, is a particular solution of Eq. [34]. Similar to the previous discussion, the equation satisfied by v_2 leads to the solution

$$v_2 = \bar{C}_{22}e^{-P_2} - ie^{-P_2}\int e^{P_2}d\tau, \qquad [35]$$

where $\bar{C}_{22}$ is a constant. Define $Q_2(\tau) = -i\int u_{p2}d\tau$, and then

$$P_2(\tau) = -i(\beta/\alpha)\sin(\alpha\tau) - 2Q_2(\tau). \qquad [36]$$

Using Eq. [35], the part of the total solution given by the reciprocal function can be expressed as

$$\frac{1}{v_2} = i\frac{d}{d\tau}\left(\log\left[\bar{C}_{22} - i\int e^{P_2}d\tau\right]\right), \qquad [37]$$

which, when substituted into the definition of y_2, gives the following expression for the solution

$$y_2 = e^{Q_2}\left[C_{21} - iC_{22}\int e^{P_2}d\tau\right], \qquad [38]$$

where $C_{21} = \bar{C}_{21}\bar{C}_{22}$ and $C_{22} = \bar{C}_{21}$. The complete solution of Eq. [34] is determined when a particular solution, u_{p2}, is found. A particular perturbation solution to Eq. [34] is determined at each order in β, by assuming

$$u_{p2} = 1 - \beta f_1 + \beta^2 f_2 - \beta^3 f_3 + \cdots, \qquad [39]$$

where only terms up to third order in β are used. It is found at order β that the equation that must be solved is the same as Eq. [18], the equation at order β^2 is identical to Eq. [21], and the equation to be solved at order β^3 is the same as Eq. [24]. Thus, the previously obtained f functions are the particular solution to Eq. [34]. Now, knowing the particular solution, it is inserted into the definition of Q_2, and it is found

$$\begin{aligned} Q_2(\tau) &= -i\int (1 - \beta f_1 + \beta^2 f_2 - \beta^3 f_3)d\tau \\ &= -iD_{00}\tau - \left(\frac{\beta}{\alpha}\right)\Bigg[-iD_1\sin\alpha\tau - D_2\cos\alpha\tau \\ &\quad + i\beta\frac{D_4}{2}\sin 2\alpha\tau + \beta\frac{D_5}{2}\cos 2\alpha\tau \\ &\quad - i\beta^2 D_6\sin\alpha\tau - \tfrac{1}{3}i\beta^2 D_7\sin 3\alpha\tau \\ &\quad - \beta^2 D_8\cos\alpha\tau - \tfrac{1}{3}\beta^2 D_9\cos 3\alpha\tau\Bigg]. \end{aligned} \qquad [40]$$

The Eq. [40] result can be substituted into Eq. [38], which leaves, as before, only one integral to be determined to get the entire solution. The integration is performed by breaking $P_2(\tau)$ into two integrand parts as

$$-i\int e^{P_2}d\tau = -i\int e^{2iD_{00}\tau}\left[e^{-i(\beta/\alpha)\sin\alpha\tau - 2Q_2 - 2iD_{00}\tau}\right]d\tau. \qquad [41]$$

The function in the square bracket in Eq. [41] is expanded to third order in β, and the integration is performed. Upon integrating Eq. [41] and substituting into Eq. [38],

$$\begin{aligned} y_2 = C_{21}e^{Q_2} + C_{22}e^{Q_2+2iD_{00}\tau}\big[&-\tfrac{1}{2} - \beta A_1\cos\alpha\tau \\ &- i\beta A_2\sin\alpha\tau + \beta^2 A_3 + \beta^2 A_4\cos 2\alpha\tau \\ &+ \beta^2 A_5\sin 2\alpha\tau - \beta^3 A_6\cos\alpha\tau - \beta^3 A_7\cos 3\alpha\tau \\ &- \beta^3 A_8\sin\alpha\tau - \beta^3 A_9\sin 3\alpha\tau\big]. \end{aligned} \qquad [42]$$

The originally desired Ψ_1 and Ψ_2 functions are obtained from Eq. [7],

$$\begin{aligned} \Psi_1 &= \frac{y_1}{2}e^{-(i\beta/2\alpha)\sin\alpha\tau} + \frac{y_2}{2}e^{(i\beta/2\alpha)\sin\alpha\tau} \\ \Psi_2 &= \frac{y_1}{2}e^{-(i\beta/2\alpha)\sin\alpha\tau} - \frac{y_2}{2}e^{(i\beta/2\alpha)\sin\alpha\tau}, \end{aligned} \qquad [43]$$

where the y_1 function is obtained from Eq. [29] and y_2 is written in Eq. [42].

7. FIRST-ORDER RESONANCE SOLUTION

The method of solution described in Section 5 relied on the determination of a particular solution of Eq. [11]. The particular solution that was determined is subject to the restriction that $\alpha \neq 2$ to prevent a singularity in the coefficients of the first-order f_1 function. It was also noted that $\alpha \neq \frac{2}{3}$ and $\alpha \neq 1$ are additional restrictions that are associated with the second- and third-order f functions. Consequently, in the first part of this section, the solution is restricted to a first-order β expansion. If α is allowed to be 2, Eq. [18] can still be solved; however, the solution is linear in τ. This is not allowed since the solution becomes unbounded. To deal with this issue, it is possible to begin again and solve for a more elaborate zero-order, particular solution of Eq. [11],

$$\frac{df_0}{d\tau} - if_0^2 = -i. \qquad [44]$$

The solution of Eq. [44] can be written with an arbitrary coefficient C_{31},

$$f_0 = \frac{1 - C_{31}e^{2i\tau}}{1 + C_{31}e^{2i\tau}}. \qquad [45]$$

The choice used in Eq. [17] was $C_{31} = 0$, which then made $f_0 = 1$. Here to avoid the singularity $C_{31} = 1$, and then $f_0 = -i \tan \tau$. As a result of this choice,

$$e^{2i \int f_0 d\tau} = \frac{1}{(\cos \tau)^2}. \qquad [46]$$

For the resonance solution, only the first-order β correction is sought and thus only f_1 is needed. The solution of Eq. [18] is written using Eq. [45] with $C_{31} = 1$,

$$\begin{aligned} f_1 &= e^{2i \int f_0 d\tau'}\left(\tfrac{1}{16} + i \int (\cos 2\tau) f_0 e^{-2i \int f_0 d\tau'} d\tau\right) \\ &= \tfrac{1}{2} \sin^2 \tau. \end{aligned} \qquad [47]$$

Going out to first order, $Q_1 = -i \int (f_0 + \beta f_1) d\tau$, and then

$$e^{Q_1} = (\cos \tau) e^{-(i\beta\tau/4) + (i\beta/8)\sin 2\tau}. \qquad [48]$$

Using Eq. [48] in Eq. [16] and keeping only terms up to first order

$$\begin{aligned} y_1 = e^{(i\beta/8)\sin 2\tau}\Bigg[& C_{11} \cos \tau e^{-(i\beta/4)\tau} - i C_{12} \\ & \times \Bigg[\sin \tau e^{(i\beta/4)\sin 2\tau + (i\beta/4)\tau} \\ & + \frac{i\beta}{16 - \beta^2} (4 \cos \tau \cos 2\tau e^{(i\beta/4)\tau}) \Bigg]\Bigg]. \end{aligned} \qquad [49]$$

As in Section 6, the functions used to determine y_1 are again used to determine y_2, thus to first order, $Q_2 = -i \int (f_0 - \beta f_1) d\tau$, and then

$$e^{Q_2} = (\cos \tau) e^{(i\beta\tau/4) - (i\beta/8)\sin 2\tau}. \qquad [50]$$

The expression of Eq. [50] for e^{Q_2} is used in Eq. [38] keeping only terms up to first order in β

$$\begin{aligned} y_2 = e^{-(i\beta/8)\sin 2\tau}\Bigg[& C_{21} \cos \tau e^{(i\beta/4)\tau} \\ & - i C_{22} \Bigg[\sin \tau e^{-(i\beta/4)\sin 2\tau - (i\beta/4)\tau} \\ & - \frac{i\beta}{16 - \beta^2} (4 \cos \tau \cos 2\tau e^{-(i\beta/4)\tau}) \Bigg]\Bigg]. \end{aligned} \qquad [51]$$

The procedure described up to this point cannot be applied to obtain a second-order accurate solution. The difficulty is that secular terms appear and the solution becomes unbounded. To gain more information about the resonance solution at higher order it is necessary to alter the formulation. This is achieved by introducing a new "time" coordinate s, that is more natural to Eq. [8]. The objective is to permit the dependent variable to contain as much of the characteristic behavior as possible. Because β is small compared to 1, the new dependent variable s is equal to the time at zero order,

$$\begin{aligned} s &= \int e^{i(\beta/2)\sin 2\tau} d\tau \\ &= \tau J_0\left(\frac{\beta}{2}\right) - i J_1\left(\frac{\beta}{2}\right)(\cos 2\tau - 1) \\ &\quad + \sum_{n=1}^{\infty} \Bigg[J_{2n}\left(\frac{\beta}{2}\right) \frac{\sin 4n\tau}{2n} \\ &\quad - i J_{2n+1}\left(\frac{\beta}{2}\right) \frac{\cos(4n+2)\tau - 1}{2n+1} \Bigg], \end{aligned} \qquad [52]$$

since the zero-order Bessel function $J(\beta/2)$ has a limit of 1 as $\beta \to 0$, and higher order Bessel functions have a limit of zero. The variable s is close to τ in its real part; however, it also has an additional imaginary part. The real part is an integral of $\cos((\beta/2)\sin 2\tau)$ and because β is always less than 1, the real part always increases as τ increases. The peculiar feature is that for a uniform increase of τ, the rate of increase of s is not uniform. The imaginary part of s is an integral of $\sin((\beta/2)\sin 2\tau)$ and because this function is odd and periodic, the imaginary part of s oscillates between a small positive and negative range having an approximate magnitude of $\beta/2$. Only a small range of s values is needed to determine s at all times, since

$$\int_{n\pi}^{\tau_0 + n\pi} e^{i(\beta/2)\sin 2\tau} d\tau = \int_0^{\tau_0} e^{i(\beta/2)\sin 2\tau} d\tau. \qquad [53]$$

In other words, if the values of s were tabulated from $\tau = 0$ to π, this would be sufficient to determine s at any value of τ.

In terms of the s variable Eq. [8] becomes

$$\begin{aligned} \frac{dy_1}{ds} &= -i y_2 \\ \frac{dy_2}{ds^*} &= -i y_1, \end{aligned} \qquad [54]$$

and the second-order equation that needs to be solved to determine y_1 is

$$\frac{d}{ds^*}\left(\frac{dy_1}{ds}\right) + y_1 = 0. \qquad [55]$$

The deceptively simple appearance of Eq. [55] results because the second derivative is with respect to the complex conjugate s^*, not just the variable s. The general form of Eq. [55] is that of an oscillator in an unusual variable space. To make further progress toward an analytic solution of y_1, Eq. [55] is converted using $d/ds^* = (ds/ds^*)(d/ds)$,

$$\frac{d^2y_1}{ds^2} + y_1\left(\frac{ds}{d\tau}\right)^{-2} = 0, \qquad [56]$$

where

$$\left(\frac{ds}{d\tau}\right)^{-2} = e^{-i\beta \sin 2\tau(s)}. \qquad [57]$$

In order to write Eq. [57] there must be a prescription to express τ as a function of s. This can be done order by order using the infinite series from Eq. [52]. To lowest order,

$$\tau_0(s) = s/J_0(\beta/2), \qquad [58]$$

and to first order in β,

$$\tau_1(s) = s/J_0(\beta/2) + iJ_1(\beta/2)\,\frac{\cos 2\tau_0(s) - 1}{J_0(\beta/2)}. \qquad [59]$$

The process can be continued as far as desired. For example, the expression for $\tau_2(s)$ would use $\tau_1(s)$ as the argument to the cosine term in Eq. [59], and then additional terms from Eq. [52] would be added.

The solution procedure for Eq. [56] begins by assuming log $y_1 = \log(C_{11}) - i \int u_1 ds$. The equation that results for u_1 is then

$$i\frac{du_1}{ds} + u^2 - 1 = \left(\frac{ds}{d\tau}\right)^{-2} - 1, \qquad [60]$$

where 1 has been subtracted from both sides to cause the lead term on the right to be order β. Based on the original first-order resonance solution, the beginning part of the solution to Eq. [60] could be $u_1 = -i\tan(s)$, since this solves the left side of Eq. [60] and is thus a zero-order solution. After a number of trial perturbation solutions it was found that a superior function results from perturbing the argument of the tangent such that $u_1 = -i\tan((s + s^*)/2 + g)$. Then rather than add perturbation terms to $-i\tan(s)$, a perturbation solution is generated for the g function. The form of the solution uses the real part of s because if just s is used, each order of g has a term which is exactly the negative of the imaginary part of s. There is no reason for g to contain a function that is known separately. Substituting $u_1 = -i\tan((s + s^*)/2 + g)$ into Eq. [60] and converting back to the variable τ results in

$$\frac{dg}{d\tau} = -i\sin\left(\frac{\beta}{2}\sin 2\tau\right)\cos(s + s^* + 2g), \qquad [61]$$

where Eq. [52] can be used to write the first few terms,

$$s + s^* = 2\tau J_0(\beta/2) + J_2(\beta/2)\sin 4\tau + (1/2)J_4(\beta/2)\sin 8\tau + \cdots. \qquad [62]$$

The benefit of solving Eq. [61] rather than Eq. [60] is that the nonlinear u^2 term is absent. It is replaced by a milder nonlinearity, in the form of the appearance of $2g$ in the argument of the cosine in Eq. [61]. The coefficient $\sin((\beta/2)\sin 2\tau)$ in Eq. [61] to lowest order is proportional to β, and thus the first-order equation becomes

$$\frac{dg_1}{d\tau} = -i\sin\left(\frac{\beta}{2}\sin 2\tau\right)\cos(2\tau). \qquad [63]$$

The solution of Eq. [63] is immediate because it was constructed to make the right side an exact derivative, and thus,

$$g_1 = -\frac{i}{\beta} + \frac{i}{\beta}\cos\left(\frac{\beta}{2}\sin 2\tau\right). \qquad [64]$$

The second-order equation,

$$\frac{dg_2}{d\tau} = i\sin\left(\frac{\beta}{2}\sin 2\tau\right)\sin(2\tau)\sin(2g_1), \qquad [65]$$

has a right-hand side that has the periodicity of $\sin 2\tau$. Furthermore, the sign of $\sin(2g_1/i)$ is always negative and the sign of $\sin((\beta/2)\sin 2\tau)\sin(2\tau)$ is always positive. As a result the sign of $dg_2/d\tau$ is always positive, and g_2 determined by Eq. [65] is not bounded. This means an order-by-order expansion solution cannot be used. Instead, Eq. [61] is viewed as an iterative solution for g, and g_1 from Eq. [64] is the starting iterate. The next iteration must solve

$$\frac{dg_2}{d\tau} = -i\sin\left(\frac{\beta}{2}\sin 2\tau\right)\cos(2\tau + 2g_1). \qquad [66]$$

The complete solution of Eq. [66] has not been obtained; however, an approximation can be written as

$$g_2 = \frac{i}{\beta}\left[\cos\left(\frac{\beta}{2}\sin\{2\tau + 2g_1\}\right) - \cos\left(\frac{\beta}{2}\cos(2\tau)\sin 2g_1\right)\right]. \qquad [67]$$

The derivative of the first cosine term on the right side of Eq. [67] produces exactly the cosine term in Eq. [66]; however, the argument of the sine differs by $2g_1$ from the desired value of 2τ. Fortunately the periodicity of $2g_1$ is given by $\sin 2\tau$, and thus it primarily acts as a small time-dependent phase shift. The argument of the second cosine term on the right side of Eq. [67] is order β^2 since $\sin 2g_1$ is order β. The derivative of $\cos((\beta/2)\cos(2\tau)\sin 2g_1)$ compensates for the inappropriate second-order terms caused by the derivative of the first cosine term, $\cos((\beta/2)\sin\{2\tau + 2g_1\})$. Each of the cosine terms in Eq. [67] spawns terms of order β^3. To go to the third iteration it would be necessary to derive terms that cancel any extraneous third-order terms generated by the iteration.

Exponentiating the relation $\log y_1 = \log(C_{11}) - i \int u_1 ds$ used to derive Eq. [60],

$$y_1 = C_{11}\cos(\tfrac{1}{2}(s + s^*) + g)e^{Q_{1c}}, \quad [68]$$

where

$$Q_{1c} = -i \int u_1 ds - \log[\cos(\tfrac{1}{2}(s + s^*) + g)]. \quad [69]$$

For convenience define $h \equiv g + \frac{1}{2}(s^* - s)$, and use $u_1 = -i \tan((s + s^*)/2 + g)$ in Eq. [69],

$$Q_{1c} = -i \int \tan(s + h)\left[1 - \frac{dh/ds}{1 + dh/ds}\right] d(s + h) - \log[\cos(\tfrac{1}{2}(s + s^*) + g)]. \quad [70]$$

Using the definition of h and $dg/d\tau$ from Eq. [61] in Eq. [70], it is found that

$$Q_{1c} = -i \int \sin(2s + 2h)\sin\left[\frac{\beta}{2}\sin 2\tau\right] d\tau. \quad [71]$$

Because Q_{1c} is order β, a second-order accurate representation results by using only g_1 in the definition of h,

$$Q_{1c} = -\frac{i\beta}{8}\left[\frac{\sin[(j_0 - 1)2\tau]}{j_0 - 1} - \frac{\sin[(j_0 + 1)2\tau]}{j_0 + 1}\right] - \frac{\beta^2}{128}\left(\frac{\cos 8\tau}{4} + \frac{\cos[(j_0 - 1)2\tau]}{j_0 - 1}\right), \quad [72]$$

where $j_0 = J_0(\beta/2)$, is a constant slightly less than 1. Since $j_0 - 1$ is small compared to 1, the first-order term, $\sin[(j_0 - 1)2\tau]/(j_0 - 1)$, is a very-low-frequency oscillation compared to the fundamental. Depending on the level of experimental resolution, the mixing of the very low frequency with the fundamental should result in a small frequency upshift and downshift, or a blurring of the resonant frequency.

The Eq. [56] governing equation for y_1 is second order, and thus it has two solutions. The second solution is derived by using $u_1 = -i \cot((s + s^*)/2 - k)$ for the solution of Eq. [60]. Substitution of this expression in Eq. [60] results in an equation for k,

$$\frac{dk}{d\tau} = -i \sin\left(\frac{\beta}{2}\sin 2\tau\right)\cos(s + s^* - 2k). \quad [73]$$

From the results and procedures used for the already obtained g function, it is found that $k_1 = g_1$ and

$$k_2 = \frac{i}{\beta}\left[\cos\left(\frac{\beta}{2}\sin\{2\tau - 2g_1\}\right) + \cos\left(\frac{\beta}{2}\cos(2\tau)\sin 2g_1\right)\right] - \frac{2i}{\beta}. \quad [74]$$

Including both solutions for u_1, using superposition, and exponentiating $\log y_1 = \log(C_{11}) - i \int u_1 ds$, which was used to derive the governing equation for u_1, yields the complete y_1 solution,

$$y_1 = C_{11}\cos(\tfrac{1}{2}(s + s^*) + g)e^{Q_{1c}} + C_{12}\sin(\tfrac{1}{2}(s + s^*) - k)e^{Q_{1s}}, \quad [75]$$

where

$$Q_{1s} = \int \cot(\tfrac{1}{2}(s + s^*) - k) ds - \log[\sin(\tfrac{1}{2}(s + s^*) - k)]. \quad [76]$$

Using the relation for $dk/d\tau$ from Eq. [73] in Eq. [76], it is found that

$$Q_{1s} = i \int \sin(s + s^* - 2k)\sin\left[\frac{\beta}{2}\sin 2\tau\right] d\tau. \quad [77]$$

As was the case for Q_{1c}, Eq. [77] is order β and, thus, a second-order accurate representation results by using only k_1 in the integrand,

$$Q_{1s} = \frac{i\beta}{8}\left[\frac{\sin[(j_0 - 1)2\tau]}{j_0 - 1} - \frac{\sin[(j_0 + 1)2\tau]}{j_0 + 1}\right] - \frac{\beta^2}{128}\left(\frac{\cos 8\tau}{4} + \frac{\cos[(j_0 - 1)2\tau]}{j_0 - 1}\right). \quad [78]$$

The corresponding function that must be derived from Eq. [54] is y_2, which is the solution of

$$\frac{d}{ds}\left[\left(\frac{ds}{ds^*}\right)\frac{dy_2}{ds}\right] + y_2 = 0. \qquad [79]$$

Using the same strategy which was applied to derive y_1, it is found that

$$y_2 = C_{21}\cos(\tfrac{1}{2}(s + s^*) + p_c)e^{Q_{2c}} + C_{22}\sin(\tfrac{1}{2}(s + s^*) - p_s)e^{Q_{2s}}, \qquad [80]$$

where $p_{s1} = p_{c1}$,

$$p_{c1} = \frac{i}{\beta} - \frac{i}{\beta}\cos\left(\frac{\beta}{2}\sin 2\tau\right), \qquad [81]$$

$$p_{c2} = \frac{i}{\beta}\left[\left(\cos\left(\frac{\beta}{2}\cos(2\tau)\sin 2p_{c1}\right) - \cos\left(\frac{\beta}{2}\sin\{2\tau + 2p_{c1}\}\right)\right)\right], \qquad [82]$$

$$p_{s2} = \frac{2i}{\beta} - \frac{i}{\beta}\left[\cos\left(\frac{\beta}{2}\sin\{2\tau - 2p_{s1}\}\right) + \cos\left(\frac{\beta}{2}\cos(2\tau)\sin 2p_{s1}\right)\right], \qquad [83]$$

$Q_{2c} = Q_{1s}$ and $Q_{2s} = Q_{1c}$.

Both y_1 in Eq. [75] and y_2 in Eq. [80] have a basic sinusoidal variation with a dominant argument of

$$\tfrac{1}{2}(s + s^*) = \tau J_0\left(\frac{\beta}{2}\right) + \sum_{n=1}^{\infty}\left[J_{2n}\left(\frac{\beta}{2}\right)\frac{\sin 4n\tau}{2n}\right]. \qquad [84]$$

This means the dominant part of the solution sine and cosine arguments depends strictly on the real part of s. From Eq. [84] it can be seen that the frequency associated with the lead linear term has been shifted by the constant $J_0(\beta/2)$. This constant is slightly less than 1 and has an expansion representation of

$$J_0\left(\frac{\beta}{2}\right) = 1 + \sum_{n=1}^{\infty}\left[\frac{(-\beta^2/16)^n}{n!n!}\right] = 1 - \frac{\beta^2}{16} + \frac{\beta^4}{1024} - \frac{\beta^6}{147456} + \frac{\beta^8}{37748736} - \cdots. \qquad [85]$$

The factorial squared denominator means the higher order terms are rapidly decreasing in magnitude. Also it can be seen that only even powers of β appear in Eq. [85], and thus, successive terms are at least smaller by a factor of $\beta^2/16$.

The argument of the cosine term in the Eq. [75] expression for y_1 is not just $(s + s^*)/2$, but g is also added to this quantity. Because the g function has not been completely determined, the obvious question is whether or not g can contribute to the coefficient of τ. In other words, can part of the solution of g have a term that is linear in τ? The answer is no, at first order, since $dg/d\tau$ as expressed in Eq. [61] is sinusoidal with no constant term. The same answer applies to k, p_c, and p_s. Thus, if g could be determined completely, it may change the coefficient of τ. The first expected consequence on the resonant frequency is a small shift downward proportional to $\beta^2/16$. This was first noted by Bloch and Siegert (*20*). There is then a finer grain modification which reduces the value of the shift by $\beta^4/1024$. This next term is smaller by a factor of $\beta^2/64$.

As mentioned earlier there are terms in, for example, Q_{1c}, which cause the resonant frequency to have a lineshape, or under extremely high resolution, a line structure. This effect is caused by an Eq. [72] term such as $-i\beta\sin[(j_0 - 1)2\tau]/[8(j_0 - 1)]$, which may be approximately written as $-2i(\sin[(j_0 - 1)2\tau])/\beta$. Because this term scales as $1/\beta$ it dominates over other Q_{1c} terms. This leads to many terms in the y_1 function that have the form of a product of sinusoids such as

$$(\cos[j_0\tau])(\cos[g])\cos\left[\frac{2}{\beta}\sin[(j_0 - 1)2\tau]\right]. \qquad [86]$$

The cosine of a sine argument can be written as an infinite sum of terms,

$$\cos\left[\frac{2}{\beta}\sin[(j_0 - 1)2\tau]\right] = J_0\left(\frac{2}{\beta}\right) + 2\sum_{n=1}^{\infty}J_{2n}\left(\frac{2}{\beta}\right)\cos[4n(j_0 - 1)\tau], \qquad [87]$$

which can be substituted into Eq. [86] to obtain

$$(\cos[j_0\tau])(\cos[g])\cos\left[\frac{2}{\beta}\sin[(j_0 - 1)2\tau]\right] = (\cos[g])\left(J_0\left(\frac{2}{\beta}\right)\cos[j_0\tau] + \Upsilon(\tau)\right), \qquad [88]$$

where

$$\Upsilon(\tau) = \sum_{n=1}^{\infty}J_{2n}\left(\frac{2}{\beta}\right)(\cos[j_0\tau + 4n(j_0 - 1)\tau] + \cos[j_0\tau - 4n(j_0 - 1)\tau]). \qquad [89]$$

The argument of the cosine terms in Eq. [89] show a $\pm 4n(j_0 - 1)$ series of shifts to the basic j_0 coefficient.

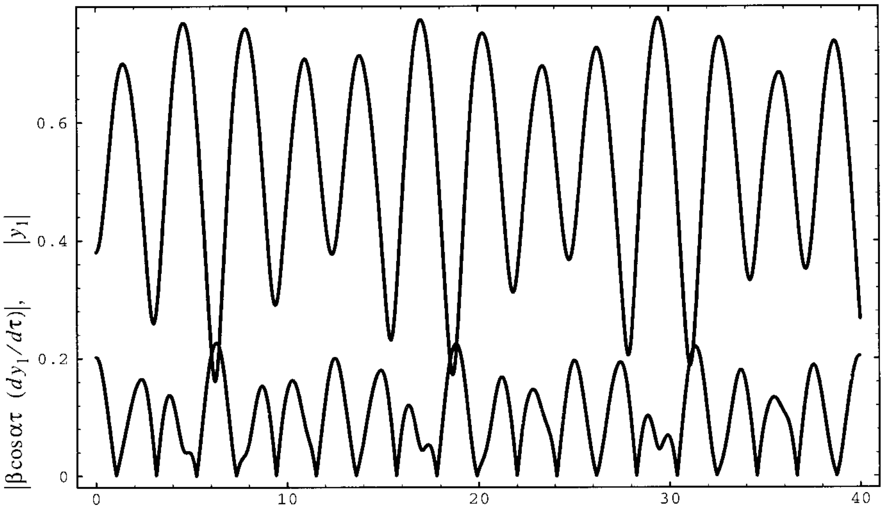

FIG. 1. Magnitude of the analytic solution y_1 for $\alpha = 1.5$ (top curve), compared to the magnitude of the perturbation $\beta\cos\alpha\tau\, dy_1/d\tau$ (bottom curve).

8. COMPARISONS BETWEEN THE ANALYTIC AND NUMERICAL SOLUTIONS

The y_1 and y_2 variables are basic to the solution since the desired spin-state probability amplitudes can be directly obtained from these functions using Eq. [43]. The governing second-order Eq. [9] can be readily solved numerically without difficulty. The disadvantage of the numerical solution is that it does not give insight into the form of the solution and must be solved repeatedly to gauge the effect of changing parameter values. On the other hand, the numerical solution can be used to evaluate the validity of the derived analytic solution. Comparisons of the numerical and analytic solutions can be made by assuming $\alpha = 1.5$, $\beta = 0.3$, $C_{11} = 0.5$, and $C_{12} = 0.5$ in the formula given by Eq. [29]. The solution to Eq. [9] is just a simple harmonic oscillator when $\beta = 0$. Furthermore, the strength of the disturbance to the harmonic solution caused by the perturbation, $-i\beta(\cos\alpha\tau)\, dy_1/d\tau$, can be observed by comparing this quantity with y_1. In Fig. 1, the magnitude of the analytic y_1 solution is plotted in the top curve and the magnitude of the perturbation is plotted in the bottom curve. It can be seen that the perturbation is about a third the size of y_1 and at times a larger fraction depending on the phasing, such as near $\tau = 6$. The real part of the analytic y_1 solution is plotted in Fig. 2 with the real part of the numerical solution. These curves are in agreement to better than 5%. The imaginary part of the analytic y_1 solution is plotted in Fig. 3 with the imaginary part of the numerical solution. These functions are nearly indistinguishable.

A second set of comparisons has been done similar to the first, with the exception that now the resonance condition $\alpha = 2$ is examined. Thus, the analytic solution which is used is given by

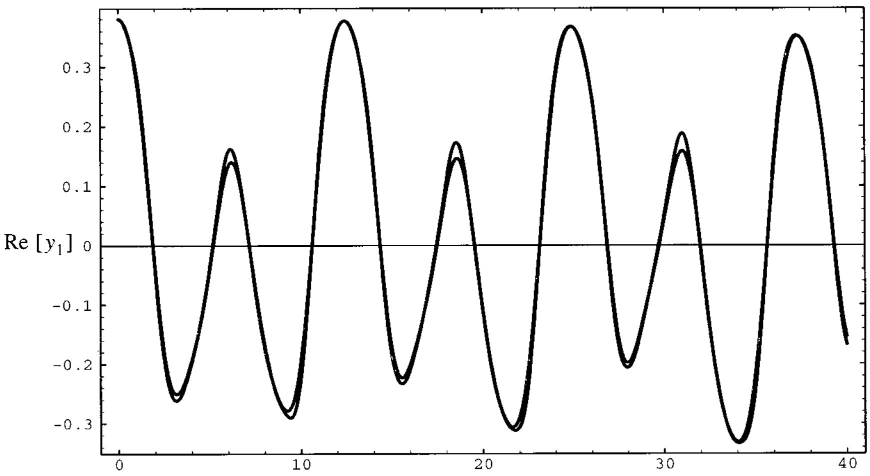

FIG. 2. Real part of the analytic solution y_1 for $\alpha = 1.5$ compared to the real part of the numerical solution.

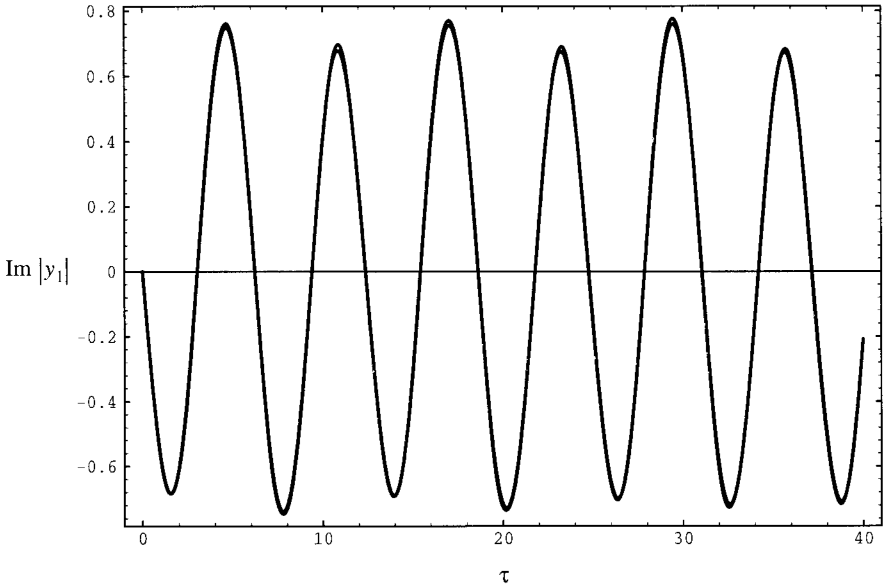

FIG. 3. Imaginary part of the analytic solution y_1 for $\alpha = 1.5$ compared to the imaginary part of the numerical solution.

the formula in Eq. [49]. In Fig. 4, the magnitude of the analytic y_1 solution is plotted in the top curve and the magnitude of the perturbation is plotted in the bottom curve. As before, the magnitude of the perturbation is about a third the size of y_1 and at times a larger fraction depending on the phasing, such as near $\tau = 8$. The real parts and imaginary parts of the analytic and numerical solutions are plotted in Figs. 5 and 6,, respectively. The agreement is good, with a 10% deviation in evidence for the peak size of the imaginary part shown in Fig. 6.

9. COMPARISON WITH PREVIOUS RESULTS

In order to compare the Ψ_1 and Ψ_2 solutions of Eq. [43] with previous results, several relationships between constants must be established. The fundamental functions, Ψ_1 and Ψ_2, have been expressed in terms of the y_1 and y_2 functions derived in Sections 5 and 6. The solution of the second-order differential equations for y_1 and y_2 each involve two arbitrary constants. Because y_1 and y_2 are related by Eq. [8] at time zero, the four constants must satisfy two coupled equations,

$$G_1C_{11} + G_2C_{12} = G_3C_{21} + G_4C_{22}$$

$$G_5C_{21} + G_6C_{22} = G_7C_{11} + G_8C_{12}. \qquad [90]$$

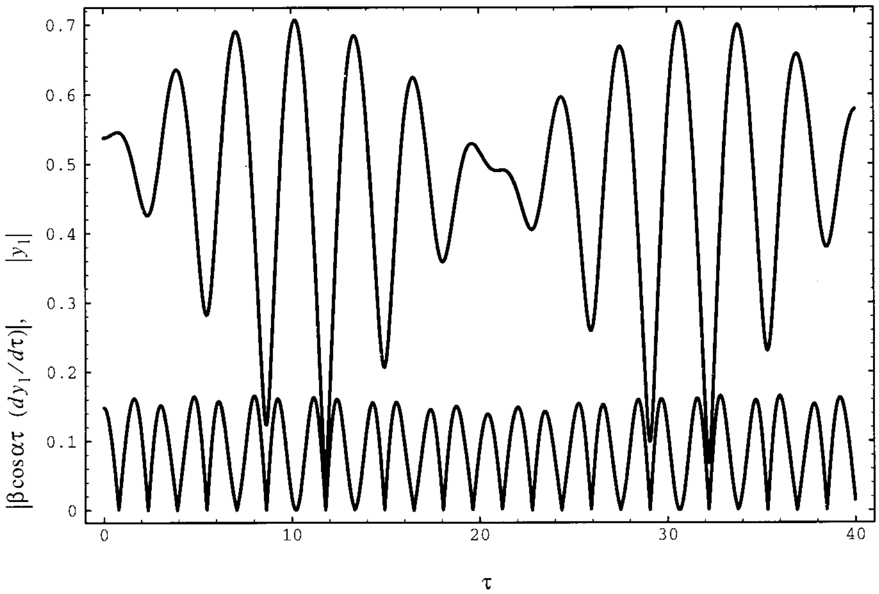

FIG. 4. Magnitude of the analytic solution y_1 for resonance at $\alpha = 2.0$ (top curve), compared to the magnitude of the perturbation $\beta \cos \alpha\tau \, dy_1/d\tau$ (bottom curve).

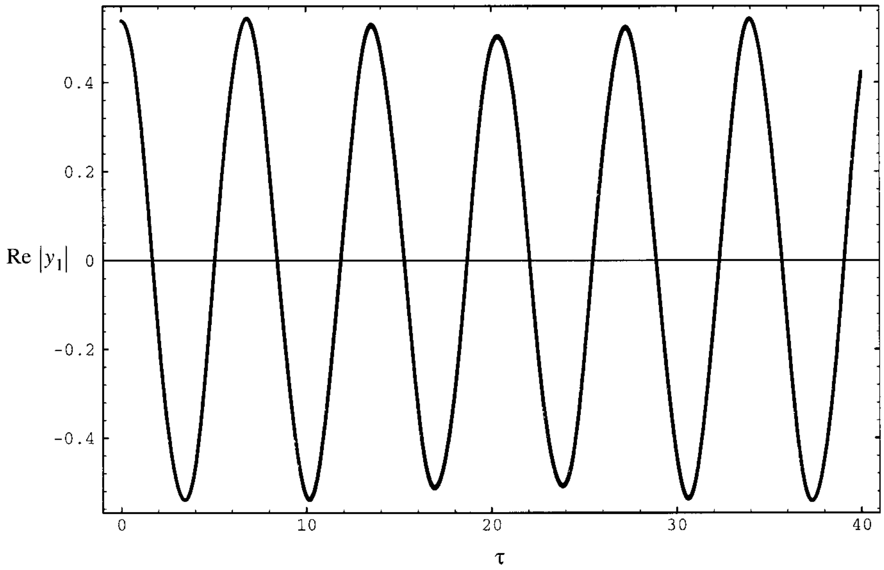

FIG. 5. Real part of the analytic solution y_1 for resonance at $\alpha = 2.0$ compared to the real part of the numerical solution.

The G_i constants in Eq. [90] only depend on α and β,

$$G_1 = e^{Q_1(0)} \frac{dQ_1}{d\tau}(0), \quad [91]$$

$$G_2 = \left(\frac{dQ_1}{d\tau}(0) + 2iD_{00}\right) e^{Q_1(0)}[-\tfrac{1}{2} + \beta A_1 + \beta^2(A_3 + A_4) + \beta^3(A_6 + A_7)] + \alpha e^{Q_1(0)}[i\beta A_2 + 2\beta^2 A_5 + \beta^3(A_8 + 3A_9)], \quad [92]$$

$$G_3 = -ie^{Q_2(0)}, \quad [93]$$

$$G_4 = G_3[-\tfrac{1}{2} - \beta A_1 + \beta^2(A_3 + A_4) - \beta^3(A_6 + A_7)], \quad [94]$$

$$G_5 = e^{Q_2(0)} \frac{dQ_2}{d\tau}(0), \quad [95]$$

$$G_6 = \left(\frac{dQ_2}{d\tau}(0) + 2iD_{00}\right) e^{Q_2(0)}[-\tfrac{1}{2} - \beta A_1 + \beta^2(A_3 + A_4) - \beta^3(A_6 + A_7)] + \alpha e^{Q_2(0)}[-i\beta A_2 + 2\beta^2 A_5 - \beta^3(A_8 + 3A_9)], \quad [96]$$

$$G_7 = -ie^{Q_1(0)}, \quad [97]$$

$$G_8 = G_7[-\tfrac{1}{2} + \beta A_1 + \beta^2(A_3 + A_4) + \beta^3(A_6 + A_7)]. \quad [98]$$

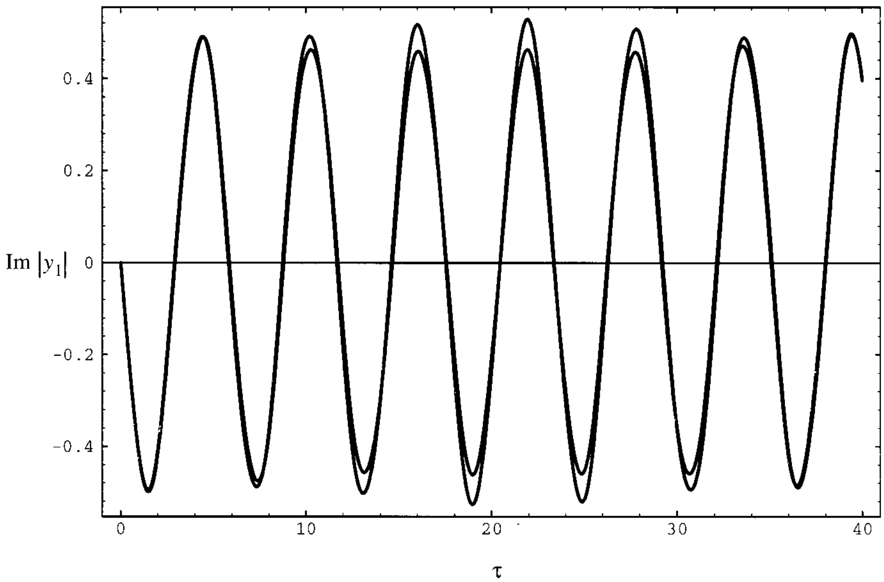

FIG. 6. Imaginary part of the analytic solution y_1 for resonance at $\alpha = 2.0$ compared to the imaginary part of the numerical solution.

From Eq. [27] at $\tau = 0$,

$$Q_1(0) = -\left(\frac{\beta}{\alpha}\right)\left[D_2 + \beta\frac{D_5}{2} + \beta^2\left(D_8 + \frac{D_9}{3}\right)\right], \quad [99]$$

and the derivative of Eq. [27] is

$$\frac{Q_1}{d\tau}(0) = -iD_{00} - i\beta[D_1 + \beta D_4 + \beta^2(D_6 + D_7)]. \quad [100]$$

From Eq. [40] at $\tau = 0$,

$$Q_2(0) = -\left(\frac{\beta}{\alpha}\right)\left[-D_2 + \beta\frac{D_5}{2} - \beta^2\left(D_8 + \frac{D_9}{3}\right)\right], \quad [101]$$

and the derivative of Eq. [40] is

$$\frac{Q_2}{d\tau}(0) = -iD_{00} - i\beta[-D_1 + \beta D_4 - \beta^2(D_6 + D_7)]. \quad [102]$$

The two relations in Eq. [90] with the initial conditions applied to Eq. [43],

$$\tfrac{1}{2}y_1(0) + \tfrac{1}{2}y_2(0) = \Psi_1(0)$$
$$\tfrac{1}{2}y_1(0) - \tfrac{1}{2}y_2(0) = \Psi_2(0), \quad [103]$$

completely define C_{11}, C_{12}, C_{21}, and C_{22} in terms of $\Psi_1(0)$ and $\Psi_2(0)$,

$$C_{11} = \frac{i\Psi_1(0)(G_8 - G_3G_{10} - G_4G_{12}) - i\Psi_2(0)(G_8 + G_3G_{10} + G_4G_{12})}{G_7(G_3G_{10} + G_4G_{12}) - G_8(G_3G_9 + G_4G_{11})}, \quad [104]$$

$$C_{12} = \frac{i\Psi_1(0)(G_7 - G_3G_9 - G_4G_{11}) - i\Psi_2(0)(G_7 + G_3G_9 + G_4G_{11})}{G_8(G_3G_9 + G_4G_{11}) - G_7(G_3G_{10} + G_4G_{12})}, \quad [105]$$

$$C_{21} = [i\Psi_1(0)(G_9(G_8 - G_4G_{12}) + G_{10}(G_4G_{11} - G_7)) - i\Psi_2(0)(G_9(G_8 + G_4G_{12}) - G_{10}(G_7 + G_4G_{11}))]/[G_7(G_3G_{10} + G_4G_{12}) - G_8(G_3G_9 + G_4G_{11})], \quad [106]$$

$$C_{22} = [i\Psi_1(0)(G_{11}(G_8 - G_3G_{10}) + G_{12}(G_3G_9 - G_7)) - i\Psi_2(0)(G_{11}(G_8 + G_3G_{10}) - G_{12}(G_7 + G_3G_9))]/[G_7(G_3G_{10} + G_4G_{12}) - G_8(G_3G_9 + G_4G_{11})], \quad [107]$$

where

$$G_9 = (G_1G_6 - G_4G_7)/(G_3G_6 - G_4G_5), \quad [108]$$
$$G_{10} = (G_2G_6 - G_4G_8)/(G_3G_6 - G_4G_5), \quad [109]$$
$$G_{11} = (G_1G_5 - G_3G_7)/(G_4G_5 - G_3G_6), \quad [110]$$
$$G_{12} = (G_2G_5 - G_3G_8)/(G_4G_5 - G_3G_6). \quad [111]$$

Thus $\Psi_1(0)$ and $\Psi_2(0)$ are the only two arbitrary constants, and the other C_{11}, C_{12}, C_{21}, and C_{22} constants associated with y_1 and y_2 depend on α, β, $\Psi_1(0)$, and $\Psi_2(0)$.

As mentioned earlier, the results of Bloch and Siegert (*20*) are related to the solutions derived in Sections 5 and 6. Their results can be compared qualitatively by considering the transition probability expression,

$$\Psi_1\Psi_1^* = \tfrac{1}{4}[y_1y_1^* + y_2y_2^* + y_1y_2^*e^{-i(\beta/\alpha)\sin\alpha\tau} + y_1^*y_2e^{i(\beta/\alpha)\sin\alpha\tau}], \quad [112]$$

where the superscript asterisk indicates a complex conjugate. The time variation of all four terms on the right side of Eq. [112] is analogous, and thus only $y_1y_1^*$ is expanded and analyzed,

$$y_1y_1^* = C_{11}C_{11}^*e^{2\mathrm{Re}[Q_1]} + C_{11}C_{12}^*e^{2\mathrm{Re}[Q_1]-2iD_{00}\tau}[y_{br}^*] + C_{12}C_{11}^*e^{2\mathrm{Re}[Q_1]+2iD_{00}\tau}[y_{br}] + C_{12}C_{12}^*e^{2\mathrm{Re}[Q_1]}[y_{br}][y_{br}^*]. \quad [113]$$

For brevity, the quantity $[y_{br}]$ in Eq. [113] is an abbreviation for the bracketed terms in Eq. [29], and

$$\mathrm{Re}[Q_1] = -\left(\frac{\beta}{\alpha}\right)\left[D_2\cos\alpha\tau + \beta\frac{D_5}{2}\cos 2\alpha\tau + \beta^2D_8\cos\alpha\tau + \frac{1}{3}\beta^2D_9\cos 3\alpha\tau\right]. \quad [114]$$

The first and fourth terms of Eq. [113] have a time variation that scales like β, which is a small parameter less than 1. The second and third terms of Eq. [113] contain the dominant time variation. Because the two terms are complex conjugates, they may be written as a magnitude, $M = 4|C_{12}C_{11}^*e^{2\mathrm{Re}[Q_1]}[y_{br}]|$ multiplying $\frac{1}{2}\cos 2(\varphi + D_{00}\tau)$. After some trigonometric algebra, the combination of the second and third terms produce $M/2 - M[\sin(\varphi + D_{00}\tau)]^2$. The $M/2$ term has a time variation that scales like β. The remaining term contains the dominant time variation and shows the sine-squared functional form evidenced in Eq. [44] of the Bloch and Siegert (*20*) result. The other prominent feature of the Bloch–Siegert result is the shift in the frequency proportional to the square of the ratio of the perpendicular magnetic field magnitude to the static magnetic field. From Section 5, $D_{00} = 1 - \beta^2/(4(4 - \alpha^2))$. Thus the dominant time variation of $\Psi_1\Psi_1^*$ has the same functional form of the Bloch and Siegert result, and the frequency shift has the same scaling.

The RWA which was discussed in the Introduction can be compared to the Section 7 resonance solution. The RWA is derived from Eq. [1] by considering the perpendicular magnetic field to be composed of two counterrotating parts and then keeping only the positive rotating part. The total perpendicular magnetic field is

$$\vec{B} = (B_1\hat{x}\cos\omega t + B_1\hat{y}\sin\omega t)/2 + (B_1\hat{x}\cos\omega t - B_1\hat{y}\sin\omega t)/2. \qquad [115]$$

The RWA keeps only the first part,

$$\vec{B} = (B_1\hat{x}\cos\omega t + B_1\hat{y}\sin\omega t)/2, \qquad [116]$$

and Eq. [1] then becomes

$$\frac{d\Psi_1}{d\tau} = -i\Psi_1 - i\left(\frac{\beta}{4}\right)\Psi_2 e^{-i\alpha\tau}$$
$$\frac{d\Psi_2}{d\tau} = i\Psi_2 - i\left(\frac{\beta}{4}\right)\Psi_1 e^{i\alpha\tau}. \qquad [117]$$

For $\alpha = 2$, the solution is

$$\Psi_1 = (\cos\tau - i\sin\tau)\left(\Psi_1(0)\cos\frac{\beta}{4}\tau - i\Psi_2(0)\sin\frac{\beta}{4}\tau\right)$$
$$\Psi_2 = (\cos\tau + i\sin\tau)\left(\Psi_2(0)\cos\frac{\beta}{4}\tau - i\Psi_1(0)\sin\frac{\beta}{4}\tau\right). \qquad [118]$$

For the Section 7 solution, the C_{ij} constants are defined in terms of initial $\Psi_1(0)$ and $\Psi_2(0)$,

$$C_{11} = \left(1 - \frac{G_{14}}{G_{13}}\right)\Psi_1(0) + \left(1 + \frac{G_{14}}{G_{13}}\right)\Psi_2(0), \qquad [119]$$

$$C_{12} = \frac{1}{G_{13}}\Psi_1(0) - \frac{1}{G_{13}}\Psi_2(0), \qquad [120]$$

$$C_{21} = \left(1 + \frac{G_{14}}{G_{15}}\right)\Psi_1(0) - \left(1 - \frac{G_{14}}{G_{15}}\right)\Psi_2(0), \qquad [121]$$

$$C_{22} = \frac{1}{G_{15}}\Psi_1(0) + \frac{1}{G_{15}}\Psi_2(0), \qquad [122]$$

where $G_{13} = 1 - 2\beta^2/(16 - \beta^2)$, $G_{14} = 4\beta/(16 - \beta^2)$, and $G_{15} = 1 + 2\beta^2/(16 - \beta^2)$. The initial values are inserted into the Section 7 expression for $\Psi_1 + \Psi_2$ to obtain

$$y_1 e^{-(i\beta/4)\sin 2\tau} = \Bigg[\Bigg(\left(\left(1 - \frac{G_{14}}{G_{13}}\right)\Psi_1(0) + \left(1 + \frac{G_{14}}{G_{13}}\right)\Psi_2(0)\right) \times \cos\tau e^{-(i\beta/8)\sin 2\tau - (i\beta/4)\tau} - i\left(\frac{1}{G_{13}}\Psi_1(0) - \frac{1}{G_{13}}\Psi_2(0)\right)\Big[\sin\tau e^{(i\beta/8)\sin 2\tau + (i\beta/4)\tau} + \frac{i\beta}{16 - \beta^2}(4\cos\tau\cos 2\tau e^{-(i\beta/8)\sin 2\tau + (i\beta/4)\tau})\Big]\Bigg]. \qquad [123]$$

The expression in Eq. [123] is to be compared to the solution derived from Eq. [118],

$$\Psi_1 + \Psi_2 = (\Psi_1(0) + \Psi_2(0))\cos\tau e^{-i(\beta/4)\tau} - i(\Psi_1(0) - \Psi_2(0))\sin\tau e^{i(\beta/4)\tau}. \qquad [124]$$

In the limit of very small β, it is approximately true that $G_{13} \approx 1$, $G_{14} \approx 0$, and $G_{15} \approx 1$, thus by comparison, the more accurate solution in Eq. [123] makes small changes to the initial condition values. However, these small changes are just sufficient so that $y_1(0) = \Psi_1(0) + \Psi_2(0)$. A $\cos\tau$ term is in both Eq. [123] and Eq. [124] but the common $e^{-i(\beta/4)\tau}$ factor has a time-varying phase modulation of size $-i(\beta/8)\sin 2\tau$. A $\sin\tau$ term also appears in both Eq. [123] and Eq. [124], but the common $e^{i(\beta/4)\tau}$ factor has a time-varying phase modulation of size $i(\beta/8)\sin 2\tau$. The largest difference between Eq. [123] and Eq. [124] is the totally new term that appears, proportional to the small parameter β. It is similar to the $\cos\tau$ term; however, it is also multiplied by $\cos 2\tau$. Because $\cos\tau\cos 2\tau = (\cos\tau + \cos 3\tau)/2$, this term can also be viewed as a further modification to the original $\cos\tau$ term plus an entirely new third harmonic contribution. The third harmonic is no longer a modification to the RWA solution in Eq. [124], but is rather a new feature which arises from including the total magnetic field of Eq. [115]. Since the phase modulation and also the third harmonic term scale as β, the difference between the Eq. [123] and Eq. [124] solutions is more apparent as the relative strength of the perpendicular time-varying magnetic field is increased.

10. CONCLUSIONS

The time evolution of the probability amplitudes of an electron in a magnetic field oriented in the z direction, perturbed by a perpendicular time-varying magnetic field, has been studied. For the situation of isolated spins, analytical expressions have been derived for the time dependence of the probability amplitudes. The intent is to apply this solution to gain an understanding of an electron beam in a manner that does not cause current loss. For example, in standard experimental practice the beam size can be determined by observing light generated by striking a material foil; however, this causes current loss and undesirable radial oscillations.

Examples of the analytic solution of the probability amplitude, for situations when the frequency of the time-varying magnetic field is 1.5 or 2.0 times the frequency associated with the steady-state magnetic field, have been compared to a numerical solution of Eq. [9]. Excellent agreement between these two solutions and the numerical solution is obtained for the first example. For the comparison with the frequency of the time-varying magnetic field twice the frequency associated with the steady-state field, the agreement is good, with a deviation between the solutions for the imaginary part of the spin-state function showing about a 10% discrepancy at the peak values. This is consistent with the derivation of the resonance solution, which is carried out to first-order in β. The next correction term would be expected to be approximately of the size β^2. For the second comparison, $\beta^{2.} = 0.09$, and this is consistent with the observed 10% discrepancy.

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